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Comparison of Global and Local Collocation Methods for Optimal Control

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I. Introduction

OPTIMAL control problems are often solved numerically via direct methods [1]. In recent years, considerable attention has been focused on a class of direct transcription methods called pseudospectral [2–4] or orthogonal collocation [5,6] methods. In a pseudospectral method, a finite basis of global interpolating polynomials is used to approximate the state at a set of discretization points. The time derivative of the state is approximated by differentiating the interpolating polynomial and constraining the derivative to be equal to the vector field at a finite set of collocation points. Although any set of unique collocation points can be chosen, generally speaking, an orthogonal collocation is chosen, i.e., the collocation points are the roots of an orthogonal polynomial (or linear combinations of such polynomials and their derivatives). Because pseudospectral methods are commonly implemented via orthogonal collocation, the terms pseudospectral and orthogonal collocation are interchangeable (thus researchers in one field may use the term pseudospectral [3], whereas others may use the term orthogonal collocation [5]). One advantage to pseudospectral methods is that for smooth problems, pseudospectral methods typically have faster convergence rates than other methods, exhibiting "spectral accuracy" [4].

Pseudospectral methods were traditionally used to solve fluid dynamics problems [3], whereas orthogonal collocation methods were first established in the chemical engineering community [6]. Seminal work in the mathematics of orthogonal collocation methods for optimal control dates back to 1979 [7] and, in recent years, the following orthogonal collocation methods have risen to prominence: the Legendre pseudospectral method [8], the Chebyshev pseudospectral method [9], the Radau orthogonal collocation method [10,11], and the Gauss pseudospectral method [12]. Within the class of pseudospectral methods, there are two very different and widely used implementation strategies that can be best described as local and global approaches. In a local approach, the time interval is divided into a large number of subintervals called segments or finite elements [6] and a small number of collocation points are used within each segment. The segments are then linked via continuity conditions on the state, the independent variable, and possibly the control. The rationale for using local collocation is that a local method provides so-called local support [13] (i.e., the discretization points are located so that they support the local behavior of the dynamics) and is both computationally simple and efficient. Although local methods have a long history in solving optimal control problems, much of the recent work has shown great success in the application of global collocation [14,15] (i.e., collocation using a global polynomial across the entire time interval). Researchers are also looking into higher-order local methods [16,17], which lie in between global and local methods. In light of the recent results that promote global orthogonal collocation and the long history of the use local collocation, it is important to gain a better understanding as to how these two different philosophies compare in practice.

This Note provides a comparison between global and local orthogonal collocation solutions for two optimal control problems. A recently developed orthogonal collocation method called the Gauss pseudospectral method (GPM) [12] is employed in both a global format and a local format. In the global approach, the GPM is implemented on a single segment and the number of collocation points is varied. As a local approach, the GPM is implemented such that a number of equal width segments are varied, while a small fixed number of collocation points are used in each segment. We note that the local application of the GPM is similar to the approaches of [6,10,18].

The results obtained in this research suggest that, except in special circumstances, global orthogonal collocation is preferable to local orthogonal collocation. For the smooth example in this study, the global GPM is much more accurate than the local GPM for a given number of total collocation points. Furthermore, for a desired accuracy, the global approach is computationally more efficient than the local approach in smooth problems. For nonsmooth problems, as in the second example, the local and global approach are quite similar in terms of accuracy.

II. Continuous Bolza Optimal Control Problem

Consider the following optimal control problem [19]. Determine the state $\boldsymbol{x}(t) \in \mathbb{R}^n$ and the control $\boldsymbol{u}(t) \in \mathbb{R}^m$ on the time interval $t \in [t_0, t_f]$ (where t_0 and t_f may be either fixed or free) that minimize the Bolza cost functional

$$J = \Phi[\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f] + \int_{t_0}^{t_f} g[\boldsymbol{x}(t), \boldsymbol{u}(t), t]\,\mathrm{d}t \tag{1}$$

subject to the dynamic constraints

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{2}$$

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the boundary conditions

$$\phi[\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f] = \boldsymbol{0} \tag{3}$$

and the inequality path constraints

$$\boldsymbol{C}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \leq \boldsymbol{0} \tag{4}$$

The problem of Eqs. (1–4) will be referred to as the *continuous Bolza problem*.

III. Gauss Pseudospectral Method

The continuous Bolza problem of Eqs. (1–4) is transcribed to a nonlinear programming problem (NLP) using the GPM [12] as follows. First, the original time interval $I = [t_0, t_f]$ is divided into S subintervals or segments I_s, $(s = 1, \ldots, S)$ such that $I_s = [t_{s-1}, t_s]$ and

$$\bigcap_{s=1}^{S} I_s = 0, \qquad \bigcup_{s=1}^{S} I_s = I$$

where $\emptyset$ is the empty set. Within each subinterval, the time $t^{(s)}$ is transformed to the interval $[-1, 1]$ via the affine transformation

$$\tau = \frac{2t^{(s)}}{t_s - t_{s-1}} - \frac{t_s + t_{s-1}}{t_s - t_{s-1}} \tag{5}$$

where $\tau \in [-1, 1]$ and denotes the transformed time in I_s. Next, assuming that the approximation has the same order in each subinterval I_s, the state is approximated using a basis of $N + 1$ Lagrange interpolating polynomials $\mathcal{L}_i(\tau)$, $(i = 0, \ldots, N)$, as

$$\boldsymbol{x}^{(s)}(\tau) \approx \boldsymbol{X}^{(s)}(\tau) = \sum_{i=0}^{N} \boldsymbol{X}^{(s)}(\tau_i)\mathcal{L}_i(\tau) \tag{6}$$

where N is the number of Legendre-Gauss (LG) points [defined as the roots of the Nth degree Legendre polynomial $P_N(\tau)$] in segment $s \in [1, \ldots, S]$. Differentiating Eq. (6) gives

$$\dot{\boldsymbol{x}}^{(s)}(\tau) \approx \dot{\boldsymbol{X}}^{(s)}(\tau) = \sum_{i=0}^{N} \boldsymbol{X}^{(s)}(\tau_i)\dot{\mathcal{L}}_i(\tau) \tag{7}$$

The derivative of the ith Lagrange polynomial at the LG points $\dot{\mathcal{L}}_i(\tau_k)$, $(k = 1, \ldots, N)$, can be represented in a differential approximation matrix $D \in \mathbb{R}^{N\times(N+1)}$ and is determined offline. The dynamic constraint is transcribed into algebraic constraints via the differential approximation matrix as

$$\sum_{i=0}^{N} D_{ki}\boldsymbol{X}_i^{(s)} - \frac{t_s - t_{s-1}}{2}\boldsymbol{f}(\boldsymbol{X}_k^{(s)}, \boldsymbol{U}_k^{(s)}, \tau_k; t_{s-1}, t_s) = \boldsymbol{0}, \qquad (k = 1, \ldots, N) \tag{8}$$

where $\boldsymbol{X}_k^{(s)} \equiv \boldsymbol{X}^{(s)}(\tau_k) \in \mathbb{R}^n$ and $\boldsymbol{U}_k^{(s)} \equiv \boldsymbol{U}^{(s)}(\tau_k) \in \mathbb{R}^m$, $(k = 1, \ldots, N)$. Note that Eq. (8) is collocated only at the LG points and *not* at the boundary points (this form of collocation differs from other well-known pseudospectral methods such as the Legendre pseudospectral method [8] and the Chebyshev pseudospectral method [9]). The final state of each segment $\boldsymbol{X}_f^{(s)}$ is also included as a variable in the NLP, where $\boldsymbol{X}_f^{(s)}$ is defined in terms of $\boldsymbol{X}_k^{(s)}$, $(k = 0, \ldots, N)$ and $\boldsymbol{U}_k^{(s)}$, $(k = 1, \ldots, N)$ via the Gauss quadrature [20]

$$\boldsymbol{X}_f^{(s)} \equiv \boldsymbol{X}_0^{(s)} + \frac{t_s - t_{s-1}}{2}\sum_{k=1}^{N} w_k \boldsymbol{f}\left(\boldsymbol{X}_k^{(s)}, \boldsymbol{U}_k^{(s)}, \tau_k; t_{s-1}, t_s\right) \tag{9}$$

The N LG points plus the initial and final point define the set of $n = N + 2$ discretization points or nodes. Also note that the control is not discretized at the boundaries. However, the boundary control can be computed via extrapolation or by an application of Pontryagin's minimum principle [21] as shown in [22]. The continuous cost function of Eq. (1) is approximated using a Gauss quadrature as

$$J = \Phi\left(\boldsymbol{X}_0^{(1)}, t_0, \boldsymbol{X}_f^{(S)}, t_f\right) + \sum_{s=1}^{S}\frac{t_s - t_{s-1}}{2}\sum_{k=1}^{N} w_k g\left(\boldsymbol{X}_k^{(s)}, \boldsymbol{U}_k^{(s)}, \tau_k; t_{s-1}, t_s\right) \tag{10}$$

where w_k are the Gauss weights associated with the LG points [3]. Next, the boundary and linkage constraints are expressed as

$$\phi\left(\boldsymbol{X}_0^{(s-1)}, t_{s-1}, \boldsymbol{X}_f^{(s)}, t_s\right) = \boldsymbol{0}, \qquad (s = 1, \ldots, S) \tag{11}$$

Furthermore, the path constraint of Eq. (4) is evaluated at the LG points as

$$\boldsymbol{C}\left(\boldsymbol{X}_k^{(s)}, \boldsymbol{U}_k^{(s)}, \tau_k; t_{s-1}, t_s\right) \leq \boldsymbol{0} \qquad (k = 1, \ldots, N; s = 1, \ldots, S) \tag{12}$$

The cost function of Eq. (10) and the algebraic constraints of Eqs. (8), (9), (11), and (12) define an NLP whose solution is an approximate solution to the continuous Bolza problem.

In the case where *one* subinterval is used (i.e., $S = 1$), the GPM is employed as a global collocation method. Conversely, in this analysis, we will take any case where $S \gg 1$, to be a local application of the method (although, strictly speaking, this is untrue). In the analysis that follows, we will use the GPM in both global collocation (i.e., $S = 1$) and local collocation (i.e., $S \gg 1$ and small values of N).

IV. Global and Local Applications of GPM

The Gauss pseudospectral method is applied as both a global and a local collocation method on two examples. As a global collocation method, the number of segments is set to unity (i.e., $S = 1$) and the accuracy is assessed as a function of the number of collocation points N. As a local collocation method, the number of collocation points in a given subinterval is fixed and the number of subintervals is varied. In this work, five nodes (three LG points plus two boundary points) are chosen for each subinterval and the width of each subinterval is constrained to be equal. Figure 1 depicts the distribution of nodes and collocation points for both the global and local approaches using a total of 20 discretization points. The LG points are denser near the endpoints than in the middle, as the nonuniformity in the LG points

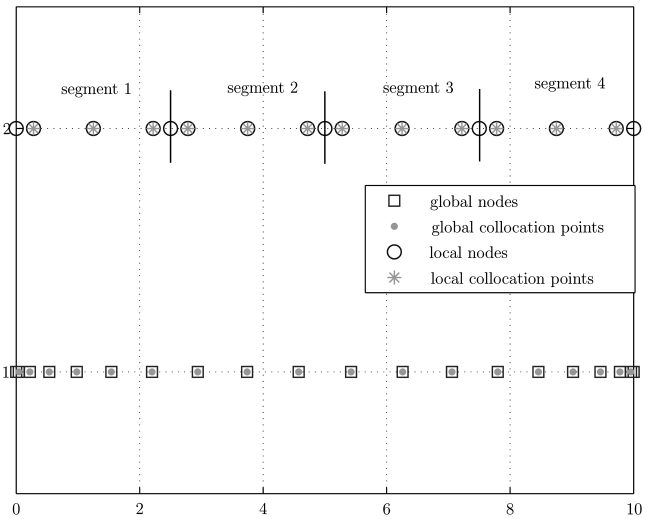

Fig. 1 Distribution of nodes and collocation points for both the global and local approaches ($N = 20$).

improves the accuracy of numerical quadratures and polynomial approximations across the interval [3]. Also note that, in a local approach, two nodes coincide at each segment interface, meaning there are $S-1$ fewer unique nodes than there are in a global approach. However, the results will show that the differences in accuracy and efficiency are significant even when comparing the number of unique nodes for each approach.

All examples were solved using the TOMLAB version [23] of the sparse NLP solver SNOPT [24] using default tolerances and comparable initial guesses for both the local and global approaches. A Pentium 4, 3.2 GHz desktop computer was used for all computations and any documented computation times are in reference to this machine.

A. Example 1: Modified Hicks–Ray Reactor Problem

Consider the Hicks–Ray reactor problem [18] where it is desired to minimize the quadratic cost functional

$$\int_0^{10} \{a_1[C(t)-\bar{C}]^2 + a_2[T(t)-\bar{T}]^2 + a_3[U(t)-\bar{U}]^2\}\,\mathrm{d}t \tag{13}$$

subject to the nonlinear dynamic constraints

$$\dot{C}(t) = \frac{1-C(t)}{\theta} - k_{10}C(t)e^{-t/T(t)} \tag{14}$$

$$\dot{T}(t) = \frac{y_f - T(t)}{\theta} + k_{10}C(t)e^{-t/T(t)} \tag{15}$$

$$-\alpha[T(t)-y_c]U(t) \tag{16}$$

and the initial conditions

$$C(0) = 0.1367 \qquad T(0) = 0.7293 \tag{17}$$

The two states are denoted by $C(t)$ and $T(t)$, the control is denoted by $U(t)$, and all other parameters are constants. This problem has no analytic solution, however, this analysis will compare the NLP solutions to a highly accurate extremal solution of the Hamiltonian boundary-value problem (HBVP) associated with Eqs. (13–17). This HBVP solution was found using the MATLAB boundary-value problem solver BVP4C [25]. It is noted that the traditional form of this example includes the (active) path constraint $U(t) \geq 0$. However, due to the increased difficulty in accurately solving a path-constrained HBVP and the fact that this is simply an illustrative example, the path constraint was omitted for this analysis.

The comparison of the Hicks–Ray reactor solution using the GPM as both a global and a local collocation method is shown in Fig. 2. The "error" in this figure represents the maximum absolute error between the NLP state and the "exact" HBVP state across the entire discretized trajectory for a given number of nodes n. For the global approach (top figure), the error decreases *spectrally* [3] as n increases, leveling off just below the feasibility tolerance of 10^{-6} at 21 nodes. This rapid convergence is typical for pseudospectral

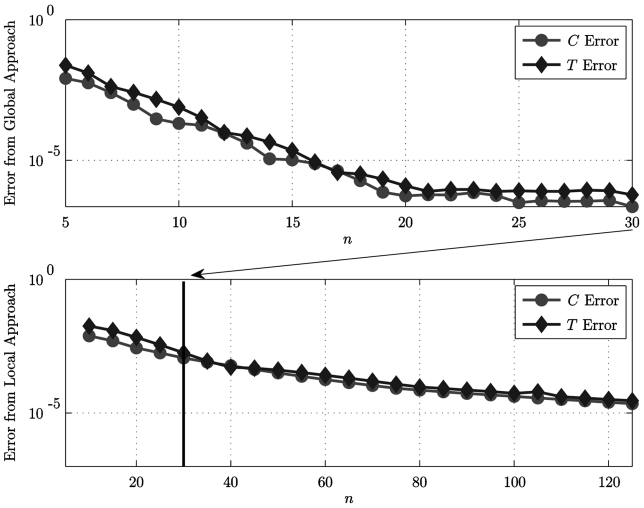

Fig. 2 Convergence of Hicks–Ray problem via both global and local approaches as a function of the number of nodes n.

methods and is one of the primary motivations for using a global approach on smooth problems. As for the local approach, Fig. 2 also shows the error in the state as the number of segments was increased (bottom figure). The errors in the state using local collocation converge at a much slower rate than seen in the global approach. Notice that the data shown for the global approach end at 30 nodes, whereas the data for the local approach extend to 125 nodes (25 segments). For comparison, at 30 nodes, the global method has an accuracy better than 10^{-6}, whereas the local method produces an accuracy of $\sim 10^{-3}$. In practice, it is common to desire a certain accuracy rather than stop at an arbitrary number of nodes. In this light, the data show that an accuracy of 10^{-5} can be obtained at 16 nodes in the global approach, whereas it takes more than 125 total nodes in the local approach.

Although the global method outperforms the local method in accuracy, one must also consider the computational efficiency between the two approaches. First, it is noted that, for the same number of nodes, the global method results in a much more dense constraint Jacobian as compared with the local method because the collocation equations are a function of the state and control across the entire interval in the global method. However, as seen from Fig. 2, if a certain accuracy is required, as is often the case, the local approach requires many more collocation points as compared with the global approach. This results in a much larger, albeit sparser, Jacobian in the local approach. Table 1 summarizes the performance of each method for a desired accuracy. For instance, Table 1 shows that, for an accuracy of 10^{-4}, the local method requires significantly more computational effort as compared with the global approach, both in terms of CPU time and major iterations of the NLP solver. This result is not intuitive, as one might expect the sparse NLP solver SNOPT to perform faster on the large, sparse approach. For this example, the extremely high accuracy of global polynomials outweighs the

Table 1 Computational expense for global and local approach

Global approach						
Accuracy	Nodes required	Nonzeros	Jacobian density, %	CPU time, s	Major iterations	Minor iterations
1e − 3	10	248	46	1.343	29	45
1e − 4	12	348	44	1.234	31	55
1e − 5	16	596	41	5.047	113	144
1e − 6	21	996	40	6.531	115	171
Local approach						
Accuracy	Nodes required	Nonzeros	Jacobian density, %	CPU time, s	Major iterations	Minor iterations
1e − 3	35	512	6.6	8.890	43	100
1e − 4	80	1178	2.8	49.86	56	174
1e − 5	165	2436	1.4	148.3	44	285

relatively inexpensive computations of the local approach in terms of computational efficiency.

B. Example 2: Double Integrator

In this problem, it is desired to minimize the cost functional

$$J = t_f \tag{18}$$

subject to the dynamic constraints

$$\dot{x}_1(t) = x_2(t) \qquad \dot{x}_2(t) = u(t) \tag{19}$$

the boundary conditions

$$x_1(0) = x_{10}; \qquad x_2(0) = x_{20} \qquad x_1(t_f) = 0; \qquad x_2(t_f) = 0 \tag{20}$$

and the inequality control constraint

$$|u(t)| \leq 1 \tag{21}$$

The key difference between this example and the previous example is that, for many initial conditions, the optimal control for this problem is "bang–bang," i.e., it switches from either its maximum value to its minimum value (or vice versa) at a time t_b (where $0 < t_b < t_f$). Because of the discontinuity in the optimal control, it would appear at first glance that a global approximation would be much less accurate than a local approximation. However, the two methods are surprisingly comparable in their solution accuracy for this problem.

Assume that no a priori knowledge of the discontinuity exists, meaning the switching time is unknown. For the initial conditions of Eq. (22), the optimal switch time of the control is $t_b \approx 4.2361$, which is designed to lie on the interior of a segment in the local approach, as opposed to on a segment boundary.

$$x_{10} = 3, \qquad x_{20} = 2 \tag{22}$$

Figure 3 shows the state error (x_1 and x_2) for both the global and local GPM solutions. The error shown here is the absolute difference between the NLP solution and exact analytic solution at each node. Naturally, the largest state error occurs at the switch time, and is strikingly similar for both approaches. The local approach is more accurate away from the discontinuity, but this is largely due to the fact that the smooth portions of the state trajectory can be approximated exactly with low-order polynomials. In general, this will not be true and the errors will be larger for the local approach in these cases. Lastly, it is noted that if a priori knowledge of the switch time does exist, a global approach can be broken into two segments. This technique has been used successfully several times using the GPM [14,15,22] or other pseudospectral methods [26–29].

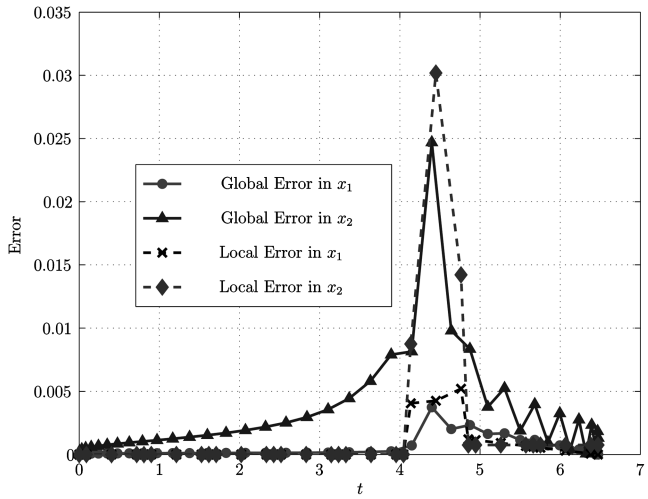

Fig. 3 Error in state for double integrator problem for both global and local LG collocation using 40 nodes.

V. Conclusions

A comparison between a global and a local implementation of an orthogonal collocation method for solving optimal control problems has been made. The purpose of the comparison was to determine the accuracy and computational efficiency of each approach. The method chosen for this comparison was the recently developed Gauss pseudospectral method. Employing the GPM as a local method, the number of collocation points within each segment was held fixed at a small number (e.g., three collocation points) while the number of segments was varied. In global collocation form, a single segment was used across the entire interval and the number of collocation points was varied. Analysis of the examples in this paper suggests that for smooth problems, global collocation is significantly more accurate than local collocation for a given number of total collocation points. For nonsmooth problems, the accuracies of global and local collocation methods were found to be comparable. Furthermore it was found that, for a desired accuracy, the global approach was computationally more efficient for smooth problems. This work supports these claims by illustrating two example problems.

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